MATHEMATICS

B. Sc, Part - II , Paper III

Real Analysis

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THEOREM Cauchy's general principle of convergence.

Statement. The necessary and sufficient condition for the convergence of any sequence $\{u_n\}$ is that corresponding to every arbitrary ϵ there exists a positive integer m such that $|u_{n+p} - u_n| < \epsilon$,
for all values of $n \geq m$ and for all positive integral values of p.

Proof The condition is necessary.

Let $u_n \to l$ as $n \to \infty$

Then $|u_n - l| < \frac{\epsilon}{2}$, when $n \geq m$.

Since $n+p > m$, therefore $|u_{n+p} - l| < \frac{\epsilon}{2}$, where p is any positive integer.

$$\therefore |u_{n+p} - u_n| = |(u_{n+p} - l) - (u_n - l)|$$
$$\leq |u_{n+p} - l| + |u_n - l|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

when $n \geq m$ and p is any positive integer.

The condition is sufficient.

Let $|u_{n+p} - u_n| < \epsilon$, when $n \geq m$ and p is any +ve integer.

Then $u_n - \epsilon < u_n + \epsilon$, when $n \geq m$ and p is any +ve integer.

Thus $\{u_{n+p}\}$ is bounded when $p \to \infty$.

If N and M be the lower and upper bounds respectively, then $N \geq u_n - \epsilon$ and $M \leq u_n + \epsilon$.

$\therefore M - N \leq u_n + \epsilon - u_n + \epsilon = 2\epsilon$.

Since ϵ is an arbitrarily small positive number, therefore

$$M - N = 0 \text{ i.e., } M = N$$
$$\therefore M - \epsilon < u_{n+p} < M + \epsilon$$

i.e. $|u_{n+p} - M| < \epsilon$

i.e $u_{n+p} M$ as $p \to \infty$

Hence $\{u_n\}$ converges to M.

THEOREM Show that the sequence $\{a_n\}$, where $a_n = \left(1+\frac{1}{n}\right)^n$, is convergent.

Proof Let us first show that the sequence is monotone ascending.

Since n is a positive integer, therefore, by the Binomial Theorem, we have

$$\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{\underline{|2}}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{\underline{|3}}\cdot\frac{1}{n^3} + \cdots + \frac{1}{n^n}$$

or $$a_n = 1+1+\frac{1}{\underline{|2}}\left(1-\frac{1}{n}\right) + \frac{1}{\underline{|3}}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots \frac{1}{\underline{|n}}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right).$$

Now changing n to $n+1$, we get

$$a_{n+1} = 1+1+\frac{1}{\underline{|2}}\left(1-\frac{1}{n+1}\right) + \frac{1}{\underline{|3}}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots + \frac{1}{\underline{|n}}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right) + \frac{1}{(n+1)^{n+1}}.$$

In a_n and a_{n+1} we find that the first two terms are equal and the third $(n-1)$ terms of a_{n+1} are each greater than the corresponding terms of a_n.

At the same time a_{n+1} contains an extra positive term $\frac{1}{(n+1)^{n+1}}$.

$\therefore a_{n+1} > a_n$, for $n = 1, 2 \cdots$

Hence the sequence $\{a_n\}$ is monotone ascending.

Now we shall prove that the sequence is bounded above.

clearly $a_n < 1+1+\frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots + \frac{1}{\underline{|n}}$

or $a_n < 1+1+\frac{1}{2}+\frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$

or $a_n < 3$.

This shows that the sequence $\{a_n\}$ is bounded above

Corollary, since $a_n < 3$, therefore $\lim_{n\to\infty} a_n < 3$.

Again $a_n > 2$; $\therefore \lim_{n\to\infty} a_n > 2$.

Hence $2 < \lim_{n\to\infty} a_n < 3$.

(1) Show that $\sqrt[n]{n} \to 1$ as $n \to \infty$.

Solution If $a_n = \sqrt[n]{n}$, let us write $a_n = 1 + b_n$, where $b_n > 0$.

Now $a_n = n^{\frac{1}{n}}$

$n = (a_n)^n$

$= (1 + b_n)^n$

$= 1 + nb_n + \frac{n(n-1)}{\lfloor 2} b_n^2 + \cdots + b_n^n$

or $n > \frac{n(n-1)}{\lfloor 2} b_n^2$; or $\frac{2}{n-1} > b_n^2$

or $b_n < \sqrt{\frac{2}{n-1}}$ $\therefore 0 < b_n < \sqrt{\frac{2}{n-1}}$.

$\therefore 0 < \lim_{n\to\infty} b_n < \lim_{n\to\infty} \sqrt{\frac{2}{n-1}}$; or $0 < \lim_{n\to\infty} b_n < 0$.

$\therefore \lim_{n\to\infty} b_n = 0$.

Hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1 + b_n) = 1$.

Thus $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(2). Prove that the sequence $\sqrt{2}, \sqrt[2]{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ is convergent, and find its limit.

Solution We have $u_1 = \sqrt{2}$ ——— (1)

Also $u_{n+1} = \sqrt{2u_n}$ ——— (2)

Putting $n=1$ in (2), we get $u_2 = \sqrt{2u_1} = \sqrt{2\sqrt{2}}$.

Evidently $\sqrt{2} > 1$; or $2\sqrt{2} > 2$; or $\sqrt{2\sqrt{2}} > \sqrt{2}$; or $u_2 > u_1$ with the help of (1).

Let $u_{n+1} > u_n$. Then $2u_{n+1} > 2u_n$; or $\sqrt{2u_{n+1}} > \sqrt{2u_n}$

or $u_{n+2} > u_{n+1}$, with the help of (2).

$\therefore u_{n+1} > u_n$ implies that $u_{n+2} > u_{n+1}$.

Hence, by mathematical induction, the given sequence $\{u_n\}$ is monotone increasing.

Now $u_1 = \sqrt{2}$. Evidently $\sqrt{2} < 2$. $\therefore u_1 < 2$.

Let $u_n < 2$. Then, by (2), $u_{n+1} = \sqrt{2u_n} < \sqrt{2 \cdot 2} = 2$.

$\therefore u_n < 2$ implies that $u_{n+1} < 2$.

Hence, by mathematical induction, $u_n < 2$, for all $n \in N$.

Therefore the sequence $\{u_n\}$ is bounded above.

Thus it must be convergent.

Let $\lim_{n\to\infty} u_n = l$

By (2), $u_{n+1} = \sqrt{2u_n}$, or $u_{n+1}^2 = 2u_n$

or $\lim_{n\to\infty} u_{n+1}^2 = 2 \lim_{n\to\infty}$; or $l^2 = 2l$

or $l = 2$, as $l \neq 0$.

$\therefore \sqrt{2} = u_1 < u_2 < u_3 < \cdots$

$\therefore \{u_n\}$ is bounded below by $\sqrt{2}$.

$\therefore \lim_{n\to\infty} u_n \geq \sqrt{2}$.

Hence $l = 2$

(3) Show that the sequence $\{x_n\}$ where $x_1 = 1$ and $x_n = \sqrt{2 + x_{n-1}}$ is convergent and converges to 2.

<u>Solution</u>: We have $x_1 = 1$ and $x_n = \sqrt{2 + x_{n-1}}$

Putting $n = 2$ in (2), we get $x_2 = \sqrt{2 + x_1} = \sqrt{2+1}$, by (1)

<u>or</u> $x_2 = \sqrt{3}$ <u>or</u> $x_2 > 1$ as $\sqrt{3} > 1$, <u>or</u> $x_2 > x_1$, by (1)

Let $x_n > x_{n-1}$. Then $2 + x_n > 2 + x_{n-1}$

<u>or</u> $\sqrt{2 + x_n} > \sqrt{2 + x_{n-1}}$ <u>or</u> $x_{n+1} > x_n$, by (2).

$\therefore$ By mathematical induction, $x_{n+1} > x_n$, for all $n \in N$.

Therefore, the sequence $\{x_n\}$ is monotone increasing.

Now $x_1 = 1$. $\therefore x_1 < 2$ as $1 < 2$

Let $x_n < 2$. Then $2 + x_n < 2 + 2$ or $\sqrt{2 + x_n} < \sqrt{4}$ or $x_{n+1} < 2$, — by (2)

$\therefore$ By mathematical induction $x_n < 2$ for all $n \in N$.

Therefore, the sequence $\{x_n\}$ is bounded above.

Hence it ~~convergent~~ must be convergent.

Let $\lim_{n\to\infty} x_n = l$.

Now, by (2), $x_n = \sqrt{2 + x_{n-1}}$; $x_n^2 = 2 + x_{n-1}$

or $\lim_{n\to\infty} x_n^2 = 2 + \lim_{n\to\infty} x_{n-1}$

or $l^2 = 2 + l$; or $l^2 - l - 2 = 0$

or $(l-2)(l+1) = 0$; $l = 2$ or -1

Since $x_n > 0$ for all $n \in N$, therefore l. that is, $\lim_{n\to\infty} x_n$ cannot be negative

$\therefore l \neq -1$.

Hence $l = 2$.